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Rotational Frames and Euclidean Connection

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Abstract

The time-dependent rotational transformation, which is a special case of the time-dependent linear transformation of coordinates in Newtonian mechanics, is considered rigorously from the point of view of infinitesimal transformation. By this approach the standard techniques in differential geometry can be naturally introduced to classical dynamics. The relation between rotational reference frames and E. Cartan's Euclidean connection is obtained. It is suggested that the extension of the present theory to the (time-dependent) general linear transformation is possible by using the bundle $L(M)$ of linear frames over a manifold M.

1. *Introduction*

Let $x \equiv (x_1, x_2, x_3)$ and $x' \equiv (x'_1, x'_2, x'_3)$ be the coordinates of the position of an identical particle observed by the two observers S and S', respectively, who are in an arbitrary relative motion in a three-dimensional Euclidean space. We regard time t as an absolute (invariant) parameter. Then the transformation $x \to x'$ can be represented as follows:

$$x \to x' = x \cdot A(t) + a(t) \tag{1.1}$$

where

$$A(t) \equiv [a_{ij}(t)], \qquad a(t) \equiv (a_1(t), a_2(t), a_3(t))$$

and

$$\det|a_{ij}(t)| = 1, \qquad {}^tA(t) = A^{-1}(t) \text{ for all } t \tag{1.2}$$

If we fix time t in these equations, the set of these transformations forms the so-called group of motions, which is the fundamental group of Euclidean geometry in F. Klein's viewpoint (Yano, 1968). In this connection we have formally discussed from the viewpoint of a finite transformation the time-dependent transformation (1.1) in the framework of Newtonian mechanics excluding the restriction of the inertial frame (Ohkuro, 1976); in this discussion two formal assumptions have been made, viz. the "vectorial property of transformation,"

$$m\frac{d^2x}{dt^2} \to m\frac{d^2x}{dt^2}\cdot A(t), \qquad F \to F\cdot A(t) \tag{1.3}$$

and the "covariance,"

$$m\frac{d^2x'}{dt^2} = F' \tag{1.4}$$

of Newton's equation of motion,

$$m\frac{d^2x}{dt^2} = F \tag{1.5}$$

under the transformation (1.1). Here m and F represent mass and force, respectively. In this way we have pointed out the importance of the time-dependent transformation (1.1) in connection with the problem of the accelerated frames. However, the assumption (1.3) is irrelevant, which can be easily seen by differentiation of equation (1.1). Therefore we need the more rigorous mathematical treatment of the transformation (1.1).

In this paper we consider the transformation (1.1) only for the case $a(t) = 0$ from the viewpoint of an infinitesimal transformation:

$$x \to x' = x\cdot A(t)$$

where

$$\det|A(t)| = 1 \text{ and } {}^tA(t) = A^{-1}(t) \text{ for all } t \tag{1.6}$$

This transformation corresponds to that to the rotational frame.

2. *Acceleration as a Tensor*

For the linear transformation (1.6), we can give its rigorous treatment as shown in this and the subsequent sections.

Let $\mathrm{d}P$ be the infinitesimal displacement of the position P of a particle in a three-dimensional Euclidean space E^3 between the time t and $t + dt$. (We use the symbol d, instead of d, for the expression of an infinitesimal vector. We use the symbol d for the exterior differentiation.) The identical quantity $\mathrm{d}P$ may be represented by two observers S and S', who are in a relative rotational motion, at time t as follows:

$$\mathrm{d}P = dx\cdot \mathrm{e} = \mathrm{d}x'\cdot \mathrm{e}' \tag{2.1}$$

where

$$dx \equiv (dx_1, dx_2, dx_3), \qquad \mathrm{e} \equiv {}^t(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$$

and the equations of the same forms for dx' and e', respectively. Here e and e' are the bases of the two orthonormal Cartesian coordinate systems. Equation (2.1) is the approximate expression (within the first order with respect to infinitesimal quantity) of the infinitesimal displacement $\mathrm{d}P(t)$ of a moving point $P(t)$ between time t and $t+dt$, i.e., $P(t+dt) - P(t) \approx \{P(t) + \mathrm{d}P(t)\} - P(t) = \mathrm{d}P(t) \equiv \mathrm{d}P$, in terms of the bases at time t: $\mathrm{e} \equiv \mathrm{e}(t)$ and $\mathrm{e}' \equiv e'(t)$. Next we specify the infinitesimal change of the bases, $\mathrm{de} \equiv \mathrm{de}(t) \approx \mathrm{e}(t+dt) - \mathrm{e}(t)$ and $\mathrm{de}' \equiv \mathrm{de}'(t) \approx \mathrm{e}'(t+dt) - \mathrm{e}'(t)$ by the equations

$$\mathrm{de} = \Omega \cdot \mathrm{e} \text{ and } \mathrm{de}' = \Omega' \cdot \mathrm{e}' \tag{2.2}$$

where the antisymmetric matrices Ω and Ω' of differential forms of degree 1 are given as follows:

$$\Omega = dT(t) = \frac{dT(t)}{dt} dt$$

and

$$\Omega' = dT'(t) = \frac{dT'(t)}{dt} dt \tag{2.3}$$

where $T(t)$ and $T'(t)$ are given by

$$\mathrm{e}(t) = T(t) \cdot \mathrm{e}(t=0)$$

and

$$\mathrm{e}'(t) = T'(t) \cdot \mathrm{e}'(t=0) \tag{2.4}$$

Here the time-dependent matrices $T(t)$ and $T'(t)$ satisfy the condition for $A(t)$ of the same form as equation (1.2). The equations (2.1)–(2.4) are the fundamental tools of our infinitesimal approach.

From these equations we can calculate the "infinitesimal displacement" of the velocity vector $\mathrm{d}P/dt$ as follows:

$$\begin{aligned} \mathrm{d}\left(\frac{\mathrm{d}P}{dt}\right) &= \mathrm{d}\left(\frac{dx}{dt} \cdot \mathrm{e}\right) \equiv \left\{d\left(\frac{dx}{dt}\right)\right\} \cdot \mathrm{e} + \frac{dx}{dt} \cdot \mathrm{de} \\ &= \left(\frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{\Omega}{dt}\right) \cdot \mathrm{e}dt \end{aligned} \tag{2.5}$$

and on the other hand we have

$$\mathrm{d}\left(\frac{\mathrm{d}P}{dt}\right) = \mathrm{d}\left(\frac{dx'}{dt} \cdot \mathrm{e}'\right) = \left(\frac{d^2x'}{dt^2} + \frac{dx'}{dt} \cdot \frac{\Omega'}{dt}\right) \cdot \mathrm{e}' \cdot dt \tag{2.6}$$

According to the condition for the observers S and S' given at the end of Section 1, we have the relation for e and e' as follows:

$$\mathrm{e}' = A(t) \cdot \mathrm{e} \tag{2.7}$$

where $A(t)$ is the one given by equation (1.6). Using this equation we have following equations:

$$\mathrm{d}P = dx \cdot \mathrm{e} = dx' \cdot \mathrm{e}' = dx' \cdot A(t) \cdot \mathrm{e}$$

$$\frac{\mathrm{d}P}{dt} = \frac{dx}{dt} \cdot \mathrm{e} = \frac{dx'}{dt} \cdot \mathrm{e}' = \frac{dx'}{dt} \cdot A(t) \cdot \mathrm{e}$$

and

$$\mathrm{d}\left(\frac{\mathrm{d}P}{dt}\right) = \left(\frac{d^2x}{dt^2} + \frac{dx}{dt}\cdot\frac{\Omega}{dt}\right)\cdot \mathrm{e}dt$$
$$= \left(\frac{d^2x'}{dt^2} + \frac{dx'}{dt}\cdot\frac{\Omega'}{dt}\right)\cdot \mathrm{e}'dt$$
$$= \left(\frac{d^2x'}{dt^2} + \frac{dx'}{dt}\cdot\frac{\Omega'}{dt}\right)\cdot A(t)\cdot \mathrm{e}dt$$

from which we have the following equations, which show the tensorial character, for the transformation (2.7), of the corresponding quantities:

$${}^t(dx') = A\cdot {}^t(dx)$$
$${}^t\left(\frac{dx'}{dt}\right) = A\cdot {}^t\left(\frac{dx}{dt}\right)$$

and

$${}^t\left(\frac{d^2x'}{dt^2} + \frac{dx'}{dt}\cdot\frac{\Omega'}{dt}\right) = A\cdot {}^t\left(\frac{d^2x}{dt^2} + \frac{dx}{dt}\cdot\frac{\Omega}{dt}\right) \tag{2.8}$$

From the last one of equations (2.8) we obtain the definition of acceleration as a tensor under the transformation (1.6): We define an acceleration tensor by the coefficient of covarient differentiation of the velocity tensor dx/dt. If Ω is independent of t, then our acceleration reduces to the usual one, d^2x/dt^2. This is the case for the fixed frame in Newtonian mechanics, because $T(t)$ is independent of t in equation (2.4) in this case.

3. *Generalization to Non-Euclidean Space*

In the previous section, (1) we have defined the "connection" of a Euclidean space (or an orthonormal Cartesian frame) *with respect to the time-variable t*, i.e., equations (2.1)-(2.4). Furthermore, (2) we have defined an acceleration-tensor for the rotational frame *in a Euclidean space.* In this section we consider the possibilities of modification and generalization of these two conditions, respectively, which will enable us to introduce the standard technique in differential geometry, i.e., the concept of fiber bundle of a differentiable manifold, to our present problem.

In the previous section we have represented the difference $\mathrm{de}(t) \approx \mathrm{e}(t+dt) - \mathrm{e}(t)$ of the basis vector between time t and $t+dt$ by the basis vector $\mathrm{e}(t)$ at time t, i.e., equation (2.2). We can apply the same technique for the coordinates x instead of time t, because we are not considering such a problem as a stocastic motion in the present paper, and because we are considering such a motion that the position $x(t+dt)$ of a particle at time

$t + dt$ is uniquely and smoothly determined by that $x(t)$ at time t. Therefore we can regard $\mathrm{e}(t + dt)$ as $\mathrm{e}(x + dx)$,

$$\mathrm{e}(t + dt) \to \mathrm{e}(x + dx) : \mathrm{e}(x + dx) \equiv \mathrm{e}[x(t) + dx(t)]$$
$$\approx \mathrm{e}[x(t + dt)] = \mathrm{e}(t + dt)$$

where

$$x(t) : t \to x \quad \text{(unique and smooth)} \tag{3.1}$$

and

$$x + dx = x(t) + dx(t) \approx x(t + dt)$$

The same modification applies also to the infinitesimal vector $\mathrm{d}P$ given by equation (2.1). Thus we obtain the "connection" of an orthonormal Cartesian frame along the path $x(t)$ of a particle regarding the time t as an implicit parameter. Hereafter we mean this modified sense when we refer to equations (2.1)-(2.4) as "connection" of a Euclidean space.

In the previous section we obtained acceleration as a tensor under the transformation of equation (1.6). The discussion given there can also be applied to the Euclidean tangent space. In fact if we require the tensorial character only of the acceleration, then the base space given by coordinates x need not be a Euclidean space. It is sufficient if its tangent space is a Euclidean space. Therefore we regard the discussion given in Section 2 as the one in the Euclidean tangent space, which has a rotational group as its fundamental group, of a general (differentiable) base manifold. Then the orthonormal Cartesian frames e and e' are interpreted as the local frames in the fiber on a point x in the base manifold.

From these considerations we arrive at the following conceptions: Let x be the coordinates of the three-dimensional differentiable base manifold M^3. Along the path $x(t)$ of a particle, where $x(t)$ is a unique and smooth function of time t, Euclidean tangent spaces $\{E^3(x(t)) | a < t < b\}$ are assigned, and the relation between $E^3(x)$ and $E^3(x + dx)$ along the path is given by the "connection" (2.1)-(2.4).

Thus the rotational motion of a particle *in a Euclidean space* can rigorously be treated using the fiber, whose fundamental group is rotation, along the path $x(t)$ in the base differentiable manifold M^3: The position of a particle can be regarded as a point in the manifold M^3, and the observers and the group of the transformation between them can be regarded as the frames in the fiber and its structure group, respectively. It is to be noted that even if we start with *a Euclidean space*, it is necessary to introduce *the manifold*, which is generally not a Euclidean space, when we discuss *the motion* in it (the Euclidean space) eliminating the transformations between observers, i.e., in a covariant fashion.

4. *Euclidean Connection*

In the preceding section the fiber along the path $x(t)$ in M^3 was introduced. We embed this fiber into the usual fiber bundle over M^3 with the rotational group as its structure group. Thus we are led to investigate the fiber bundle itself, where the bundle space is obtained by moving the point P over M^3 from the set F_P of the system of all positive orthonormal vectors in the tangent space at a point P. In this section we investigate the same problem from the classical viewpoint of E. Cartan, i.e., the Euclidean connection in a differentiable manifold. [Nowadays E. Cartan's Euclidean connection is called a metric connection of Riemannian manifold (Kobayashi and Nomizu, 1963).] We proceed according to Yano (1968). First let (x^1, x^2, x^3) be the coordinates of a point of a general three-dimensional space M^3. Next we suppose that the tangent space at each point P of M^3 is the space with the group of motions as its structure group, that is, a Euclidean space in Klein's sense. We assign linearly independent three tangent vectors $A_j(x), j = 1, 2, 3$ to each point $P(x), x \in M$, of M^3. [$A_j(x), j = 1, 2, 3$ need not be an orthogonal system.] Then the point $P(x + dx) \approx P + \mathrm{d}P$ in the neighborhood of the point $P(x)$ is expected to be described in the form

$$P + \mathrm{d}P = P + \sum_{j=1}^{3} \omega^j A_j \tag{4.1}$$

because the point $P(x + dx)$ can be regarded as the point in the tangent space at the point $P(x)$. Here ω^i are Pfaffian forms depending on the coordinates x^i. ω^i satisfy the equation of the following form:

$$\omega^j = \sum_{i=1}^{3} p^j{}_i dx^i \tag{4.2}$$

where $p^j{}_i$ are differentiable functions of x. Thus we have the expression for the infinitesimal vector $\mathrm{d}P$

$$\mathrm{d}P = \sum_{j=1}^{3} \omega^j A_j \tag{4.3}$$

We have the equations

$$\mathrm{d}P = \sum_j \left(\sum_i p^j{}_i dx^i \right) A_j = \sum_i \left(\sum_j p^j{}_i A_j \right) dx^i = \sum_i A_i dx^i \tag{4.4}$$

where we write $\Sigma_j p^j{}_i A_j$ as A_i again.

When we put this tangent Euclidean space at the point $P + \mathrm{d}P$ upon the other tangent Euclidean space at the point P, we must assign the positions which the vectors $A_j + \mathrm{d}A_j$ in the tangent space at the point $P + \mathrm{d}P$ take in the other tangent space at the point P; we must assign the vectors $A_j + \mathrm{d}A_j$ using the tangent vectors A_j. (Here it should be noted that we are *not* assuming the orthogonality, but assuming the linear independence of vectors A_j in the tangent Euclidean space.) This can be done by the equation

$$A_j + \mathrm{d}A_j = A_j + \sum_{i=1}^{3} \omega_j^i A_i \tag{4.5}$$

i.e.,

$$\mathrm{d}A_j = \sum_{i=1}^{3} \omega_j^i A_i \tag{4.6}$$

where the quantity $\omega_j{}^i$ is a Pfaffian form depending on the coordinates x, and we define the connection coefficients Γ^i_{jk} by the equation

$$\omega_j{}^i = \sum_{k=1}^{3} \Gamma^i_{jk} dx^k \tag{4.7}$$

Finally the Euclidean connection of our space is represented by the following formulas:

$$\mathrm{d}P = A_i dx^i \tag{4.8}$$

and

$$\mathrm{d}A_j = \Gamma^i_{jk} dx^k A_i \tag{4.9}$$

where both here and hereafter repeated indices are summed over. The Euclidean connection of our space M^3 is determined by assignment of $3^3 = 27$ functions Γ^i_{jk}. Generally Γ^i_{jk} is not symmetric: $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

Let us consider the transformation of coordinates

$$x \to \bar{x} \tag{4.10}$$

The above formulas can be written in the coordinates $\bar{x}$ as follows:

$$\mathrm{d}\bar{P} = \bar{A}_i d\bar{x}^i \tag{4.11}$$

and

$$\mathrm{d}\bar{A}_j = \bar{\Gamma}^i_{jk} d\bar{x}^k \bar{A}_i \tag{4.12}$$

In the former the infinitesimal vector $\mathrm{d}P$ is equal to the other infinitesimal vector $\mathrm{d}\bar{P}$,

$$\mathrm{d}P = \mathrm{d}\bar{P} \tag{4.13}$$

because the point P is invariant for the transformation of coordinates

$$P = \bar{P} \tag{4.14}$$

Therefore from equations (4.8) and (4.11) we have

$$\bar{A}_i = \frac{\partial x^a}{\partial \bar{x}^i} A_a \tag{4.15}$$

which shows that A_i are the components of a covariant vector. We have the equations, using equation (4.15),

$$\begin{aligned}
\mathrm{d}\bar{A}_j &= \mathrm{d}\left(\frac{\partial x^a}{\partial \bar{x}^j} \cdot A_a\right) = d\left(\frac{\partial x^a}{\partial \bar{x}^j}\right) \cdot A_a + \frac{\partial x^a}{\partial \bar{x}^j} \cdot \mathrm{d}A_a \\
&= \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} d\bar{x}^k A_a + \frac{\partial x^a}{\partial \bar{x}^j} \cdot \mathrm{d}A_a \\
&= \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} \cdot d\bar{x}^k A_a + \frac{\partial x^a}{\partial \bar{x}^j} \cdot \Gamma^i_{ab} A_i \frac{\partial x^b}{\partial \bar{x}^k} \cdot d\bar{x}^k
\end{aligned} \tag{4.16}$$

where in the last equality we have used equation (4.9). On the other hand from equations (4.12) and (4.15) we have the equations

$$d\bar{A}_j = \bar{\Gamma}^i_{jk} d\bar{x}^k \bar{A}_i = \bar{\Gamma}^i_{jk} \frac{\partial x^a}{\partial \bar{x}^i} A_a d\bar{x}^k \tag{4.17}$$

From equations (4.16) and (4.17) we have

$$\frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} + \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} \Gamma^a_{bc} = \bar{\Gamma}^i_{jk} \frac{\partial x^a}{\partial \bar{x}^i} \tag{4.18}$$

which gives the transformation rule of the connection coefficient for the transformation of coordinates. (See also Yano, 1968).

The torsion form τ and curvature form Θ are introduced according to Flanders (1963) as follows:

$$\begin{aligned} d^2P &= d(dP) = d(A_i dx^i) = dA_i \cdot dx^i + A_i d^2 x^i \\ &= dA_i \cdot dx^i = \omega_i^j A_j \cdot dx^i = \Gamma^j_{ik} dx^k dx^i A_j \\ &\equiv \tau^j A_j = \tau \cdot A \end{aligned} \tag{4.19}$$

where

$$\tau \equiv (\tau^1, \tau^2, \tau^3) \text{ and } A \equiv {}^t(A_1, A_2, A_3) \tag{4.20}$$

The torsion coefficient $T^i{}_{jk}$ is defined by the equation

$$\tau^i = \tfrac{1}{2} T^i{}_{jk} dx^j dx^k, \qquad T^i{}_{jk} = -T^i{}_{kj} \tag{4.21}$$

Therefore we have the equation

$$T^i{}_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj} \tag{4.22}$$

On the other hand we have the equation

$$\begin{aligned} d^2A_i &= d(dA_i) = d(\omega_i^j A_j) = d\omega_i^j \cdot A_j - \omega_i^j dA_j \\ &= d\omega_i^j \cdot A_j - \omega_i^j \omega_j^k \cdot A_k = (d\omega_i^j - \omega_i^k \omega_k^j) \cdot A_j \\ &\equiv \theta_i^j \cdot A_j = \Theta \cdot A \end{aligned} \tag{4.23}$$

where

$$\Theta \equiv (\theta_i^j)$$

Thus we have the equation

$$\Theta = d\omega - \omega^2 \tag{4.24}$$

where

$$\omega \equiv (\omega_i^j)$$

The curvature tensor $R_i{}^j{}_{kl}$ is defined by the equation

$$\theta_i^j = \tfrac{1}{2} R_i{}^j{}_{kl} dx^k dx^l, \qquad R_i{}^j{}_{kl} = -R_i{}^j{}_{lk} \tag{4.25}$$

In equations (4.19)-(4.25) products of differential forms represent the exterior products. From equation (4.23) we have the equation

$$R_i{}^j{}_{kl} = \frac{\partial \Gamma^j_{il}}{\partial x^k} - \frac{\partial \Gamma^j_{ik}}{\partial x^l} + \Gamma^m_{il}\Gamma^j_{mk} - \Gamma^m_{ik}\Gamma^j_{ml} \tag{4.26}$$

It is well known that the space with Euclidean connection is a generalization of Riemannian space in the sense that in the former torsion form is generally not zero whereas in the latter it is usually set identically equal to zero; Riemannian geometry is based on the torsionless connection. In the space with Euclidean connection the base manifold M^3 can be locally approximated by the tangent *Euclidean* space whose basis is given by the set of linearly independent covariant vectors (A_1, A_2, A_3).[1] Therefore we can consider the length ds of the infinitesimal vector $\mathrm{d}P = A_i dx^i$:

$$\begin{aligned} ds^2 &= \mathrm{d}P \cdot \mathrm{d}P = (A_j \,.\, dx^j)(A_k \cdot dx^k) \\ &= A_j \cdot A_k dx^j dx^k \end{aligned} \tag{4.27}$$

where the product $dx^j dx^k$ is not the exterior product. Thus we have

$$ds^2 = g_{jk} dx^j dx^k \tag{4.28}$$

where

$$g_{jk} = A_j \cdot A_k, \qquad g_{jk} = g_{kj} \tag{4.29}$$

Equation (4.28) shows that our space is a Riemannian space. (Only our choice of connection is different from the case in conventional Riemannian geometry.) Differentiating equation (4.29) we have the equations

$$(\mathrm{d}A_j) \cdot A_k + A_j \cdot (\mathrm{d}A_k) = dg_{jk}$$

and

$$(\Gamma^a_{jh} dx^h A_a) \cdot A_k + A_j \cdot (\Gamma^a_{kh} dx^h A_a) = dg_{jk}$$

Therefore because of the arbitrariness of dx^h we have

$$\Gamma^a_{jh} g_{ak} + \Gamma^a_{kh} g_{ja} = \frac{\partial g_{jk}}{\partial x^h} \tag{4.30}$$

which is the condition to be satisfied by the functions Γ^i_{jk} in order that the given connection Γ^i_{jk} be nothing but a Euclidean connection (Yano, 1968). Nowadays a connection satisfying equation (4.30) is called a metric connection of a Reimannian space. In particular, Christoffel's symbol $\{^{\,i}_{jk}\} = \{^{\,i}_{kj}\}$, i.e., torsion-

[1] A_i need not be "orthonormal" here. On this point Euclidean connection can be regarded as a preliminary step to a linear connection in the principal fiber bundle $L(M)$ of linear frames over M, which corresponds to the generalization of equation (1.6) such that $A(t)$ is an element of the general linear transformation group $GL(3; \mathbf{R})$ of dimension 3 for any fixed t. In particular Euclidean connection is a connection in $O(M)$, i.e., the principal fiber bundle of orthonormal frames over M (Kobayashi and Nomizu, 1963). See also Section 5.

less connection, satisfies equation (4.30), as is well known, and therefore $\{{}^{i}_{jk}\}$ is one of the solutions of equation (4.30). However, the values of Γ^{i}_{jk} satisfying equation (4.30) exist innumerably besides $\{{}^{i}_{jk}\}$ (Yano, 1968), where we have

$$\{{}^{i}_{jk}\} = \{{}^{i}_{kj}\} = \tfrac{1}{2} g^{ia}\left(\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a}\right) \tag{4.31}$$

and

$$g^{ia} g_{aj} = \delta^i_j = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j \end{cases} \tag{4.32}$$

5. *Discussion and Conclusion*

Even if we restrict ourselves to the rotational motion of a particle in a Euclidean space, from the viewpoint of the transformation of coordinates the mathematically rigorous treatment is not so simple as the conventional discussion in classical dynamics. It is necessary to introduce the differentiable manifold M^3 with a tangent Euclidean space, i.e., *the fiber bundle on* M^3 with the rotational group as its structure group, in order to discuss even the rotational motion in a *Euclidean space.* The position of a particle can be regarded as a point in the base manifold M^3, and the observers and the group of transformation between them can be regarded as the frames in the fiber and its structure group, respectively. This corresponds to the principal fiber bundle $O(M)$ of orthonormal frames over M.

In Section 4 we have given a brief explanation of the conception of Euclidean connection according to Yano. Euclidean (or metric) connection is the one that has the character between Riemannian connection (or Levi-Civita connection) and affine connection (or linear connection). It is to be noted that in the space with Euclidean connection we have the metric tensor g_{ij} as well as the torsion form; in Riemannian connection the torsion form is zero, and in affine connection we cannot define the concept of the length of a vector (therefore we cannot define the metric g_{ij} either) (Yano, 1968).

In Riemannian geometry the Ricci tensor R_{jk}, where

$$R_{jk} \equiv \sum_a R_j{}^a{}_{ak} \tag{5.1}$$

plays an important role. However, in the space with Euclidean connection the quantity corresponding to this contraction does not have any geometrical (or invariant) meaning for transformations of frames, as seen from equation (4.25). On the other hand the trace of the curvature from Θ, which is invariant under the transformation of frames, has the following form:

$$\operatorname{Tr} \Theta = \sum_i \theta_i{}^i = \frac{1}{2} \sum_{k,l} \left(\sum_i R_i{}^i{}_{kl} \right) dx^k dx^l \tag{5.2}$$

Thus in our approach the quantity $\Sigma_i R_i{}^i{}_{kl}$ must have an important meaning instead of the "*Ricci tensor*" $\Sigma_a R_j{}^a{}_{ak}$. In fact we can give the new definition of the gravitational field (different from Einstein's theory) in terms of the quantity $\Sigma_i R_i{}^i{}_{kl}$ instead of $\Sigma_a R_j{}^a{}_{ak}$. Then we can obtain the right-hand side, the force term, of our new equation of motion using the quantity $\Sigma_i R_i{}^i{}_{kl}$, in which the acceleration tensor is given by equation (2.8). The detailed discussion of these points will be given in a separate paper, in which the results given in the present paper will be extensively used.

It is to be noted that in Riemannian geometry we have the trivial result

$$\sum_i R_i{}^i{}_{kl} \equiv 0 \tag{5.3}$$

which is another reason that we must extend Riemannian geometry, or rather Riemannian connection, itself. Riemannian connection is a connection in the principal fiber bundle $O(M)$ of orthonormal frames over M with a corresponding metric given by equations (4.28) and (4.29). In equation (4.29) the inner product $A_j \cdot A_k$ of two tangent vectors A_j and A_k means that of two vectors in a Euclidean space. Equation (5.3) holds even in a Euclidean connection of M, if we use the metric given by equations (4.28) and (4.29). The reason is as follows: The metric admits the choice of orthonormal frames of tangent space $T_x(M)$ at $x \in M$, and therefore the curvature matrix $\Theta \equiv (\theta_i{}^j)$ becomes antisymmetric so that

$$\mathrm{Tr}\,\Theta = 0$$

To use equation (5.2) we must generalize the metric given by equations (4.28) and (4.29); this leads us to the concept of the principal fiber bundle $L(M)$ of linear frames over M. This corresponds to an extension of $A(t)$ in equation (1.6) to $GL(3; \mathbf{R})$. This subject will be discussed in a separate paper.

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